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Between Quasi-Convex and Convex Set-Valued Mappings

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Abstract—The aim of this paper is to give sufficient conditions for a quasi-convex set-valued mapping to be convex. In particular, we recover several known characterizations of convex real-valued functions, given in terms of quasiconvexity and Jensen-type convexity by Nikodem [1], Behringer [2], and Yang, Teo and Yang [3]. © 2004 Elsevier Ltd. All rights reserved.

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1. INTRODUCTION

Throughout the paper, we denote by X a linear space and by Y a topological linear space, partially ordered by a closed convex cone K having a nonempty interior in Y. Let $F : D \to 2^Y$ be a set-valued mapping, defined on a nonempty convex subset D of X.

Recall that F is said to be *K-convex* if the inclusion

$$tF(x) + (1-t)F(x') \subset F(tx + (1-t)x') + K \tag{1}$$

holds for all $x, x' \in D$ and for every $t \in [0,1]$.

By analogy to vector-valued functions, we say that F is *K-quasiconvex* if for each $y \in Y$ the level set $L_F(y) := \{x \in D : y \in F(x) + K\}$ is convex. Since K is a convex cone, it can be easily seen that F is K-quasiconvex whenever it is K-convex.

In order to get sufficient conditions for a K-quasi-convex mapping to be K-convex, we shall consider the following concept of generalized convexity: F will be called *weakly K-convex with respect to a nonempty set $T \subset]0,1[$* if for all $x, x' \in D$ there exists some $t \in T$ for which (1) holds.

Note that this concept extends several notions of generalized convexity, which were intensively studied in the literature in the particular case of real-valued functions. Indeed, if $T \subset]0,1[$ is a

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singleton, we recover the Jensen-type convexity (see, e.g., [4] and references therein), which is nowadays also known as nearly convexity (see, e.g., [5]). On the other hand, for $T =]0,1[$ we recover the notion of weakly convexity, introduced by Aleman in [6]. As an intermediate case, if $T = [\delta, 1-\delta]$ with $\delta \in]0, 1/2[$, we recover the notion of uniform convexlikeness, which has been introduced by Hartwig in [7].

Our aim here is to study the set-valued mappings, but our main result also focuses on vector-valued functions. Actually, if $f : D \to Y$ is a function defined on a nonempty convex subset D of X, then f will be called K-convex (respectively, K-quasiconvex, or weakly K-convex with respect to a nonempty set $T \subset]0,1[$) if and only if the set-valued mapping $F : D \to 2^Y$, defined by $F(x) = \{f(x)\}$ for all $x \in D$, is K-convex (respectively, K-quasiconvex, or weakly K-convex with respect to T).

2. MAIN RESULT

THEOREM 2.1. *Let $F : D \to 2^Y$ be a set-valued mapping defined on a nonempty convex subset D of X. If F has K-closed values (i.e., $F(x) + K$ is a closed set for every $x \in D$), then the following assertions are equivalent:*

(i) *F is K-convex;*
(ii) *F is both K-quasiconvex and weakly K-convex with respect to a nonempty compact set $T \subset]0,1[$.*

PROOF. Obviously (i) implies (ii), the conclusion being true for any nonempty set $T \subset]0,1[$.

Conversely, suppose that (ii) holds and let T be a nonempty compact subset of $]0,1[$ for which F is weakly K-convex. Let us denote, for all $x, x' \in D$,

$$T_{x,x'} := \{t \in [0,1] : (1) \text{ holds}\}.$$

In order to prove (i), we just have to show that $T_{x,x'} = [0,1]$ for all $x, x' \in D$. To this end, consider two arbitrary points $x_0, x_1 \in D$ and let us first prove that T_{x_0,x_1} is dense in $[0,1]$. Suppose on the contrary that this is not the case. Then there exist some $a, b \in [0,1]$, $a < b$, such that

$$[a,b] \cap T_{x_0,x_1} = \emptyset. \tag{2}$$

Since $\{0,1\} \subset T_{x_0,x_1}$, we can define the real numbers

$$\alpha := \sup[0,a] \cap T_{x_0,x_1} \qquad \text{and} \qquad \beta := \inf[b,1] \cap T_{x_0,x_1}. \tag{3}$$

Obviously, $\alpha \le a < b \le \beta$ and, by (2) and (3), we have

$$]\alpha, \beta[\cap T_{x_0,x_1} = \emptyset. \tag{4}$$

Let us denote, for all $t \in [0,1]$,

$$x_t := tx_0 + (1-t)x_1 \qquad \text{and} \qquad Y_t := tF(x_0) + (1-t)F(x_1).$$

By (3) and taking into account that T is compact, we can find some numbers $u \in [0,\alpha] \cap T_{x_0,x_1}$ and $v \in [\beta, 1] \cap T_{x_0,x_1}$ such that $tu + (1-t)v \in]\alpha, \beta[$ for all $t \in T$. On the other hand, since F is weakly K-convex with respect to T, we can choose a number $\tau \in T_{x_u,x_v} \cap T$. Hence,

$$\gamma := \tau u + (1-\tau)v \in]\alpha, \beta[. \tag{5}$$

Since $1 - \gamma = \tau(1-u) + (1-\tau)(1-v)$, by definition of Y_γ we can deduce that

$$\begin{aligned} Y_\gamma &= [\tau u + (1-\tau)v]F(x_0) + [\tau(1-u) + (1-\tau)(1-v)]F(x_1) \\ &\subset \tau[uF(x_0) + (1-u)F(x_1)] + (1-\tau)[vF(x_0) + (1-v)F(x_1)] \\ &= \tau Y_u + (1-\tau)Y_v. \end{aligned} \tag{6}$$

On the other hand, since $u, v \in T_{x_0,x_1}$, we have $Y_u \subset F(x_u)+K$ and $Y_v \subset F(x_v)+K$, and hence,

$$\tau Y_u + (1-\tau)Y_v \subset \tau F(x_u) + (1-\tau)F(x_v) + K. \tag{7}$$

Recalling that $\tau \in T_{x_u,x_v}$, i.e., $\tau F(x_u)+(1-\tau)F(x_v) \subset F(\tau x_u + (1-\tau)x_v)+K$, by (6) and (7) we infer that $Y_\gamma \subset \tau Y_u + (1-\tau)Y_v \subset F(\tau x_u + (1-\tau)x_v) + K = F(x_\gamma)+K$, which means that $\gamma \in T_{x_0,x_1}$. By (5) it follows that $]\alpha,\beta[\cap T_{x_0,x_1} \neq \emptyset$, contradicting (4).

So, we have proved that T_{x_0,x_1} is dense in $[0,1]$. Now, let us show that $T_{x_0,x_1}=[0,1]$. Obviously, $\{0,1\} \subset T_{x_0,x_1} \subset [0,1]$. Consider an arbitrary $t \in]0,1[$. We just need to prove that $t \in T_{x_0,x_1}$, i.e., $Y_t \subset F(x_t)+K$. Let $y \in Y_t$. By definition of Y_t we have $y = tz_0 + (1-t)z_1$ for some $z_0 \in F(x_0)$ and $z_1 \in F(x_1)$. Consider a point $e \in \operatorname{int} K$. By density of T_{x_0,x_1} in $[0,1]$, we infer the existence of two sequences: $(t_n^-)_{n\in\mathbb{N}}$ in $T_{x_0,x_1} \cap [0,t]$ and $(t_n^+)_{n\in\mathbb{N}}$ in $T_{x_0,x_1}\cap[t,1]$, such that

$$\{y_n^-, y_n^+\} \subset y + \frac{1}{n}e - \operatorname{int} K, \qquad \text{for all } n \geq 1,$$

where $y_n^- = t_n^- z_0 + (1-t_n^-)z_1$ and $y_n^+ = t_n^+ z_0 + (1-t_n^+)z_1$. Then, we have

$$y + \frac{1}{n}e \in t_n^- F(x_0) + (1-t_n^-)F(x_1) + \operatorname{int} K \subset F\left(x_{t_n^-}\right) + K + \operatorname{int} K \subset F\left(x_{t_n^-}\right) + K,$$

$$y + \frac{1}{n}e \in t_n^+ F(x_0) + (1-t_n^+)F(x_1) + \operatorname{int} K \subset F\left(x_{t_n^+}\right) + K + \operatorname{int} K \subset F\left(x_{t_n^+}\right) + K,$$

implying that $\{x_{t_n^-}, x_{t_n^+}\} \subset L_F(y+(1/n)e)$, for all $n \geq 1$. Recalling that F is K-quasiconvex and taking into account that $x_t \in [x_{t_n^-}, x_{t_n^+}]$ for each $n \in \mathbb{N}$, we can deduce that

$$x_t \in L_F\left(y + \frac{1}{n}e\right), \qquad \text{i.e.,} \qquad y + \frac{1}{n}e \in F(x_t) + K, \qquad \text{for all } n \geq 1.$$

Finally, by letting $n \to \infty$, we infer that $y \in \overline{F(x_t)+K} = F(x_t)+K$. ∎

COROLLARY 2.2. *Let $f : D \to Y$ be a function defined on a nonempty convex subset D of X. Then f is K-convex if and only if it is both K-quasiconvex and weakly K-convex with respect to a nonempty compact set $T \subset]0,1[$.*

PROOF. It follows by Theorem 2.1, where $F : D \to 2^Y$ is defined by $F(x) = \{f(x)\}$ for all $x \in D$. In this case $F(x)+K$ is closed for every $x \in D$, since the cone K is closed. ∎

REMARK 2.3. The compactness of T is essential. Indeed, consider $X = Y = \mathbb{R}$ and $K = \mathbb{R}_+$, and let $f : D = [0,1] \to \mathbb{R}$ be defined by $f(x) = 1$ if $x \in [0,1/2]$, and $f(x) = 0$ if $x \in]1/2,1]$. Then f is both quasiconvex and weakly convex with respect to $T =]0,1[$, but f is not convex.

REMARK 2.4. Corollary 2.2 generalizes some known characterization theorems given for real-valued convex functions, such as

(a) Proposition 3 in [1], where $X = \mathbb{R}^n$, $Y = \mathbb{R}$, $K = \mathbb{R}_+$, $D \subset \mathbb{R}^n$ is a nonempty convex open set, and $T = \{1/2\}$;
(b) Theorem 2 in [2], where X is a linear space, $Y = \mathbb{R}$, $K = \mathbb{R}_+$, $D \subset X$ is a nonempty convex set, and $T = \{1/2\}$;
(c) Theorem 3 in [3], where $X = \mathbb{R}^n$, $Y = \mathbb{R}$, $K = \mathbb{R}_+$, $D \subset \mathbb{R}^n$ is a nonempty convex set, and $T = \{\alpha\}$ with $\alpha \in]0,1[$.

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